MOTION OF TWO INTERACTING CHARGED PARTICLES IN A HOMOGENEOUS MAGNETIC FIELD

(O DVIZHENII DVUKH VZAIMODEISTVUIUSHCHIKH ZARIAZHENNYKH CHASTITS V ODNORODNOM MAGNITNOM POLE)

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Iu. Ia. POLIAK
(Sverdlovsk)

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Synge's monograph [1] and Rein's paper [2] contain a very general method for reducing the order of a system of Hamilton equations on the basis of a known integral. This method is an effective means of dealing with the nonrelativistic problem of the motion of two interacting charged particles in a homogeneous stationary magnetic field. In the present paper the four derived integrals of motion (in addition to the energy integral) are used to reduce this problem to one concerning the relative motion of one of the particles.

1. The Lagrange function of a system of two interacting charged particles in a homogeneous magnetic field is of the form

$$L = \frac{m_1 \mathbf{r}_1^{\cdot 2}}{2} + \frac{m_2 \mathbf{r}_2^{\cdot 2}}{2} + \frac{e_1}{2c}(\mathbf{r}_1 \times \mathbf{r}_1^{\cdot}) \cdot \mathbf{B} + \frac{e_2}{2c}(\mathbf{r}_2 \times \mathbf{r}_2^{\cdot}) \cdot \mathbf{B} - \frac{e_1 e_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \tag{1.1}$$

In the coordinates of the center of mass and of relative motion

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}, \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{1.2}$$

the Lagrange function can be written as

$$L = {}^1/_2 M \mathbf{R}^{\cdot 2} - 2M\omega_1 Y X^{\cdot} + M\omega_2 (xY^{\cdot} - yX^{\cdot}) + + {}^1/_2 m \mathbf{r}^{\cdot 2} + m\omega_3 (xy^{\cdot} - yx^{\cdot}) - \frac{e_1 e_2}{r} \tag{1.3}$$

Here

$$m = \frac{m_1 m_2}{m_1 + m_2}, \qquad M = m_1 + m_2, \qquad \omega_1 = \frac{(e_1 + e_2) B}{2c (m_1 + m_2)} \tag{1.4}$$

$$\omega_2 = \frac{(e_1 m_2 - e_2 m_1) B}{c (m_1 + m_2)^2}, \qquad \omega_3 = \frac{(e_1 m_2^2 + e_2 m_1^2) B}{2c (m_1 + m_2) m_1 m_2}$$

The magnetic field has a z-component only. Lagrangian (1.3) appears in a form asymmetrical with respect to the coordinates X and Y. This renders the x- coordinate of the center of mass cyclical. By adding the total derivative with respect to time $2M\omega_1 (YX^{\cdot} + XY^{\cdot})$, to Lagrangian (1.3), we can make Y cyclical instead of X.

The fact that (1.3) contains a term with ω_2 indicates that the motion of the center of mass and the relative motion are gyroscopically linked, while the term $2M\omega_1 YX^{\cdot}$ indicates that one of the coordinates of the center of mass is not cyclical. From (1.3) we see that in addition to the trivial case in which there is no magnetic field and a two-body problem results, the variables are also immediately separable in the two particular cases where $\omega_1 = 0$ or $\omega_2 = 0$. For $\omega_2 = 0$ the problem breaks down into two independent problems: that of the motion of the center of mass and the problem of relative motion. Solution of the former is elementary, while the latter, by virtue of cylindrical symmetry, becomes a two-dimensional problem apparently solvable by approximate methods only. For $\omega_1 = 0$ all three coordinates of the center of mass are cyclical, so that the corresponding momenta are conserved. In this case the motion of the center of mass and the relative motion remain linked, however, since $\omega_2 \neq 0$.

In order to reduce the problem to a one-particle problem (i.e. to a sixth-order Hamilton system) in the general case, it is necessary to convert from Lagrange function to the Hamilton function

$$H = \frac{1}{2M}\{(P_x + 2M\omega_1 Y + M\omega_2 y)^2 + (P_y - M\omega_2 x)^2 + P_z^2\} +$$
$$+ \frac{1}{2m}\{(p_x + m\omega_3 y)^2 + (p_y - m\omega_3 x)^2 + p_z^2\} + \frac{e_1 e_2}{\sqrt{x^2+y^2+z^2}} \tag{1.5}$$

and to make use of the motion integrals

$$P_x = P_{0x}, \qquad P_{0y} = P_y + 2M\omega_1 X, \qquad P_z = P_{0z} \tag{1.6}$$

$$y p_x - x p_y + Y P_x - X P_y + M\omega_1 (Y^2 - X^2) = L_0 \tag{1.7}$$

The validity of (1.6) and (1.7) can be verified by writing out the corresponding Hamilton functions. The known integrals can be used to find a canonical transformation as a result of which the motion integral becomes a generalized momentum and the corresponding generalized coordinate turns out to be cyclical. Following the method presented in [1 and 2], one can carry out the canonical transformations in either a Cartesian or a cylindrical coordinate system.

2. In the Cartesian coordinate system of (1.5) the x- and z-coordinates of the center of mass are cyclical. Hence, it is first necessary to eliminate P_x and X from (1.7) using (1.6) to obtain the integral

$$F_1(P_y, p_x, p_y, Y, x, y) \equiv \frac{P_y^2}{2M} + 2M\omega_1^2\left(Y + \frac{P_{0x}}{2M\omega_1}\right)^2 + 2\omega_1(y p_x - x p_y) = \text{const} \tag{2.1}$$

The generating function W_1, which depends on the previous coordinates Y, x, y, z and on the new coordinates τ, q_1, q_2, q_3 must satisfy Equation

$$\frac{\partial W_1}{\partial \tau} + F_1\left(\frac{\partial W_1}{\partial Y}, \frac{\partial W_1}{\partial x}, \frac{\partial W_1}{\partial y}, Y, x, y\right) = 0 \tag{2.2}$$

Instead of solving Hamilton-Jacobi equation (2.2) with F_1 as the Hamilton function (τ plays the role of time), we can find W_1 by solving the corresponding system of Hamilton equations and computing the principal Hamilton function. By expressing it in terms of the previous coordinates and their initial values for $\tau = 0$, we can find the generating function W_1. One of the constants of integration over τ should then be considered the parameter of the problem, and the remaining q_1, q_2, q_3 as the new coordinates (along with τ). The generating function then assumes the form

$$W_1 = (xq_2 - yq_1)\sin 2\omega_1\tau + (xq_1 + yq_2)\cos 2\omega_1\tau + zq_3 +$$
$$+ \frac{M\omega_1}{\sin 2\omega_1\tau}\left\{\left[q_0^2 + \left(Y + \frac{P_{0x}}{2M\omega_1}\right)^2\right]\cos 2\omega_1\tau - 2q_0\left(Y + \frac{P_{0x}}{2M\omega_1}\right)\right\} \tag{2.3}$$

where q_0 is a constant. The relationship between the new variables and the initial ones is given by the relations (2.4)

$$P_y = \frac{2M\omega_1}{\sin 2\omega_1\tau}\left\{\left(Y + \frac{P_{0x}}{2M\omega_1}\right)\cos 2\omega_1\tau - q_0\right\} \qquad p_z = q_3, \qquad p_3 = -z$$

$$p_x = q_1 \cos 2\omega_1\tau + q_2 \sin 2\omega_1\tau, \qquad p_y = -q_1 \sin 2\omega_1\tau + q_2 \cos 2\omega_1\tau$$

$$p_1 = -x \cos 2\omega_1\tau + y \sin 2\omega_1\tau, \qquad p_2 = -x \sin 2\omega_1\tau - y \cos 2\omega_1\tau$$

$$P_\tau = 2\omega_1 [(yq_1 - xq_2)\cos 2\omega_1\tau + (xq_1 + yq_2)\sin 2\omega_1\tau] + \\ + \frac{2M\omega_1^2}{(\sin 2\omega_1\tau)^2}\left\{\left[q_0^2 + \left(Y + \frac{P_{0x}}{2M\omega_1}\right)^2\right] - 2q_0\left(Y + \frac{P_{0x}}{2M\omega_1}\right)\cos 2\omega_1\tau\right\}$$

The following condition is fulfilled in addition to (2.2): the determinant consisting of the mixed second partial derivatives of W_1 with respect to the new and initial coordinates is not equal to zero.

Exchanging the roles of the new momenta and coordinates, we can write Hamiltonian (1.5) in the form

$$H = \frac{1}{2m}\{(p_1 + m\omega_3 q_2)^2 + (p_2 - m\omega_3 q_1)^2 + p_z^2\} + \frac{e_1 e_2}{\sqrt{q_1^2 + q_2^2 + z^2}} + \\ + \frac{1}{2M}\{[M\omega_2 q_1 \pm \sqrt{2MP_\tau - (2M\omega_1 q_0)^2 + 4M\omega_1(q_1 p_2 - q_2 p_1)}]^2 + \\ + [M\omega_2 q_2 + 2M\omega_1 q_0]^2\} \qquad (2.5)$$

The generalized momentum P_τ is here conserved and the problem has thus been reduced to a one-particle problem. From (2.4) it follows that q_1, q_2 in (2.5) have the same meaning as x, y, since they are related by a rotation transformation relative to the z-axis,

$$x = q_1 \cos 2\omega_1\tau + q_2 \sin 2\omega_1\tau, \quad y = -q_1 \sin 2\omega_1\tau + q_2 \cos 2\omega_1\tau \qquad (2.6)$$

Here the dependence of τ on time is given by

$$\dot\tau = \partial H / \partial P_\tau$$

where H is given by (2.5).

For the above particular cases $\omega_1 = 0$ and $\omega_2 = 0$ Hamiltonian (2.5) becomes simple: it lacks the square root of the expression containing $q_1 p_2 - q_2 p_1$. For $\omega_1 = 0$ from (2.6) we have $x = q_1$, $y = q_2$ and (2.5) coincides with (1.5) if we set

$$P_x = 2M\omega_1 q_0, \quad \mp P_y = \sqrt{2MP_\tau}$$

in Hamiltonian (1.5).

If $\omega_2 = 0$ (here $\omega_1 = \omega_3$), then (2.5) differs from that part of Hamiltonian (1.5) which with $\omega_2 = 0$ describes the relative motion only in the sign of ω_3. This is due to conversion to coordinate system (2.6) rotating with the constant angular velocity $2\omega_1$ in the direction opposite to that of cyclotronic rotation ($\tau = t$ for $\omega_2 = 0$).

In the absence of a magnetic field (i.e. when $\omega_1 = \omega_2 = \omega_3$), Expression (2.5) becomes the ordinary Hamiltonian of the Kepler problem.

3. In the cylindrical coordinate system

$$X = R\cos\psi, \quad Y = R\sin\psi, \quad x = \rho\cos\varphi, \quad y = \rho\sin\varphi \qquad (3.1)$$

it is important to replace the angles ψ, φ by the new angles

$$\alpha = \varphi - \psi, \qquad \beta = \varphi + \psi \tag{3.2}$$

and to make use of a Lagrangian differing from (1.3) by the total derivative with respect to time

$$M\omega_1 (XY' + YX')$$

This Lagrangian, being symmetrical in X, Y and x, y, has the two cyclical coordinates β and Z in variables (3.2). The corresponding Hamilton function is of the form

$$H = \frac{1}{2M}\left\{(P_\rho + M\omega_2\rho\sin\alpha)^2 + \left(\frac{p_\alpha - p_\beta}{R} + M\omega_1 R + M\omega_2\rho\cos\alpha\right)^2 + P_z^2\right\} + \\ + \frac{1}{2m}\left\{p_\rho^2 + p_z^2 + \left(\frac{p_\alpha + p_\beta}{\rho} - m\omega_3\rho\right)^2\right\} + \frac{e_1e_2}{\sqrt{\rho^2+z^2}} \tag{3.3}$$

It can be shown that $p_\beta = L_0$ (see (1.7)). After β has been eliminated integrals (1.6) yield

$$F_2(P_\rho, p_\alpha, R) \equiv \frac{1}{2M}\left\{P_\rho^2 + \left(\frac{p_\alpha - L_0}{R} - M\omega_1 R\right)^2\right\} = \text{const} \tag{3.4}$$

As above, we can find the generating function

$$W_2 = \frac{M\omega_1}{2\sin\omega_1\tau}\{(R^2 + R_0^2)\cos\omega_1\tau - 2RR_0\cos(\alpha - q_6 + \omega_1\tau)\} + \\ + L_0(\alpha - q_6) + \rho q_4 + zq_5 \tag{3.5}$$

Here τ, q_4, q_5, q_6 are new coordinates and R_0 is a constant. The function W_2 satisfies Equation

$$\frac{\partial W_2}{\partial\tau} + F_2\left(\frac{\partial W_2}{\partial R}, \frac{\partial W_2}{\partial\alpha}, R\right) = 0 \tag{3.6}$$

The relationship between the old and new variables is given by the relations

$$\begin{aligned} P_\rho &= M\omega_1\{R\cos\omega_1\tau - R_0\cos(\alpha - q_6 + \omega_1\tau)\}(\sin\omega_1\tau)^{-1} \\ p_\alpha &= L_0 - 2M\omega_1 R_0 R\sin(\alpha - q_6 + \omega_1\tau)(\sin\omega_1\tau)^{-1} \\ P_\tau &= {}^1/_2\, M\omega_1^2\{R^2 + R_0^2 - 2RR_0\cos(\alpha - q_6)\}(\sin\omega_1\tau)^{-2} \\ p_\rho &= q_4, \; p_z = q_5, \quad p_4 = -\rho, \quad p_5 = -z, \quad p_6 = p_\alpha \end{aligned} \tag{3.7}$$

Exchanging the roles of p_4, q_4 and p_5, q_5 in this case, we can write Hamilton function (3.3) in the new variables, using the previous notation for the angle (α instead of q_6)

$$H = \frac{1}{2m}\left\{p_\rho^2 + p_z^2 + \left(\frac{p_\alpha + L_0}{\rho} - m\omega_3\rho\right)^2\right\} + \frac{e_1e_2}{\sqrt{\rho^2+z^2}} + \\ + \frac{1}{2M}\left\{\left[M\omega_2\rho\cos\alpha + \frac{p_\alpha - L_0}{R_0} + M\omega_1 R_0\right]^2 + \left[M\omega_2\rho\sin\alpha \pm \\ \pm\left(2MP_\tau - \left[\frac{p_\alpha - L_p}{R_0} + M\omega_1 R_0\right]^2\right)^{1/2}\right]\right\} \tag{3.8}$$

Here @ is the integral of motion.

In order to compare (2.5) and (3.8) it is necessary to convert to the cylindrical coordinate system in (2.5) and to apply the canonical transformation with the generating function

$$W_3 = \alpha p_\varphi + \int_{p_\lambda}^{p_\varphi} \sin^{-1}\frac{2M\omega_1 q_0}{\sqrt{2MP_\tau + 4M\omega_1 p_\varphi}}\,dp_\varphi + \rho' p_\rho + z' p_z + \tau' P_\tau \tag{3.9}$$

Here the polar angle φ is related to the new angle α by Expression

$$\alpha = \varphi - \sin^{-1}\frac{2M\omega_1 q_0}{\sqrt{2MP_\tau + 4M\omega_1 p_\varphi}} \tag{3.10}$$

while the remaining variables (except τ) are not transformed, so that instead of (2.5) we have (omitting the primes)

$$H = \frac{p_\rho^2 + p_z^2}{2m} + \frac{1}{2m}\left(\frac{p_\alpha}{\rho}\right)^2 + (2\omega_1 - \omega_3)\, p_\alpha + \frac{M\omega_2^2 + m\omega_3^2}{2}\rho^2 +$$
$$+ P_\tau + \omega_2\rho\cos\alpha\,\sqrt{2MP_\tau + 4M\omega_1 p_\alpha} + \frac{e_1 e_2}{\sqrt{\rho^2 + z^2}} \tag{3.11}$$

For τ' from (3. 9) we have

$$\tau' = \tau - \frac{1}{2\omega_1}\left\{\sin^{-1}\frac{2M\omega_1 q_0}{\sqrt{2MP_\tau + 4M\omega_1 p_\varphi}} - \sin^{-1}\frac{2M\omega_1 q_0}{\sqrt{2MP_\tau + 4M\omega_1 p_0}}\right\}\frac{1}{2\omega_1} \tag{3.12}$$

Returning to (3. 8), let us transform this Hamiltonian by means of the generating function

$$W = \pm\int_{p_0}^{p_\alpha}\cos^{-1}\frac{p_\alpha - L_0 + M\omega_1 R_0^2}{R_0\sqrt{2MP_\tau + 4M\omega_1(p_\alpha - L_0)}}\,dp_\alpha + \alpha'(p_\alpha + L_0) + \rho' p_\rho + z' p_z +$$
$$+ \tau'(P_\tau - 4\omega_1 L_0) \tag{3.13}$$

Here

$$P_\tau' = P_\tau - 4_\omega,\ L_0, \quad p_\alpha' = p_\alpha + L_0$$

$$\alpha' = \alpha \mp \cos^{-1}\frac{p_\alpha - L_0 + M\omega_1 R_0^2}{R_0\sqrt{2MP_\tau + 4M\omega_1(p_\alpha - L_0)}} \tag{3.14}$$

and the remaining variables (except τ) are not transformed.

As a result of this transformation, (3. 8) yields Hamiltonian function (3. 11) which was previously derived from (2. 5). We have thus succeeded in reducing the twelfth-order system of Hamilton equations to a sixth-order system, i. e. to the problem of relative motion. For $\omega_2 \neq 0$ this motion is related to the motion of the center of mass by way of P_τ in (3. 11) or P_τ, q_0 in (2. 5) and P_τ, L_0, R_0 in (3. 8) .

4. , The effect of the center-of-mass motion on the relative motion has as one of its results the fact that the systems of canonical equations corresponding to Hamiltonians (2. 5), (3. 8), or (3. 11) have a time-independent solution if $\omega_2 \neq 0$.

Making use of (3. 11), we can write the equations of motion in the form

$$mz^{\cdot\cdot} = \frac{e_1 e_2 z}{(\rho^2 + z^2)^{3/2}}, \qquad p_\alpha^{\cdot} = \omega_2\rho\sin\alpha\sqrt{2MP_\tau + 4M\omega_1 p_\alpha} \tag{4.1}$$

$$m\rho^{\cdot\cdot} = \frac{1}{m\rho}\left(\frac{p_\alpha}{\rho}\right)^2 - (m\omega_3^2 + M\omega_2^2)\,\rho - \omega_2\cos\alpha\sqrt{2MP_\tau + 4M\omega_1 p_\alpha} + \frac{e_1 e_2 \rho}{(\rho^2 + z^2)^{3/2}} \tag{4.2}$$

$$\alpha^{\cdot} = \frac{p_\alpha}{m\rho^2} + 2\omega_1 - \omega_3 + \frac{2M\omega_1\omega_2\rho\cos\alpha}{\sqrt{2MP_\tau + 4M\omega_1 p_\alpha}} \tag{4.3}$$

$$\tau^{\cdot} = 1 + \frac{M\omega_2\cos\alpha}{\sqrt{2MP_\tau + 4M\omega_1 p_\alpha}} \tag{4.4}$$

This system has the trivial solution $\rho^{\cdot} = p_\alpha^{\cdot} = \alpha^{\cdot} = z^{\cdot} = z = 0$ and $\alpha = 0$ or $\alpha = \pi$. The values of @ and ρ are determined by equations (4. 2) and (4. 3) with zero as their left sides . This solution can be illustrated by means of a concrete example . In fact, (1. 6), (2. 4), (2. 6), (3. 10) and (4. 4) imply that the particles in this case are always situated at fixed points on a straight line which rotates about one of its points with the coordinates $P_{0y}/2\ M\omega_1,\ -P_{0x}/\ 2\ M\omega_1$ with the constant angular velocity

$$-2\omega_1 \pm 2M\omega_1\omega_2\rho\ (2\ MP_\tau + 4\ M\omega_1 p_\alpha)^{-1/2}$$

and with the initial phase

$$\sin^{-1} \frac{2M\omega_1 q_0}{\sqrt{2MP_\tau + 4M\omega_1 p_0}}$$

Here p_α, ρ are trivial solutions of (4, 2) and (4, 3). We note that for $e_1,\ e_2 > 0$ the solution is "longitudinally" unstable, which follows from the equation for z (4, 1).

Let us analyze the case where $\omega_1 = 0$ $(e_1 = -e_2 = -e < 0)$ in more detail. Here $p_\alpha = m\omega_3\rho^2$ and the equilibrium value of ρ is given by Equation

$$-e^2/\rho - M\omega_2^2\rho - \omega_2\sqrt{2MP_\tau} = 0 \tag{4.5}$$

In order to consider the existence and stability of the time-independent solution it is convenient to convert to the Lagrange function

$$E = \frac{m}{2}(x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2}) + m\omega_3(xy^\cdot - yx^\cdot) + \frac{e^2}{\sqrt{x^2+y^2+z^2}} -$$
$$- \frac{1}{2M}\{(M\omega_2 x - P_{0y})^2 + (M\omega_2 y + P_{0x})^2\} \tag{4.6}$$

and to investigate the behavior of equipotential surfaces of the form

$$U(x, y, z) \equiv \frac{M\omega_2^2}{2}[(x-x_0)^2 + y^2] - \frac{e^2}{\sqrt{x^2+y^2+z^2}} \tag{4.7}$$

making use of the energy integral. In contrast to (4, 6), in (4, 7) we chose a coordinate system in which $P_{0x} = 0,\ x_0 = P_{0y}/M\omega_2 > 0$. Here x_0 has a simple physical meaning in the scattering problem : it is the distance in the xy plane between the centers of the Larmor circles for $t = -\infty$ (i. e. a kind of impact parameter). The latter is easily proved by substituting into integral (1, 6) written in the form ($\omega_1 = 0$)

$$m_1 y_1^\cdot + m_2 y_2^\cdot + M\omega_2(x_1 - x_2) = P_{0y}$$

the solution for noninteracting particles in a magnetic field which is valid for the scattering problem for $t = -\infty$.

To begin with, we can show that Equation

$$U(x, 0, 0) \equiv \frac{M\omega_2^2}{2}(x-x_0)^2 - \frac{e^2}{|x|} = E \tag{4.8}$$

has one negative and three positive solutions if

$$\frac{4}{27}\frac{M\omega_2^2 x_0^3}{e^2} > 1, \qquad U_1 < E < U_2 \tag{4.9}$$

where

$$U_1 = -{}^2/_3 M\omega_1^2 x_2^2\ [1 - \cos{}^1/_3(\pi - \theta)]\cos{}^1/_3(\pi - \theta) \leqslant 0 \tag{4.10}$$

$$U_2 = -{}^2/_3 M\omega_2^2 x_0^2\ [1 - \cos{}^1/_3(\pi + \theta)]\cos{}^1/_3(\pi + \theta) \tag{4.11}$$

$$\cos\theta = -1 + \frac{27}{2}\frac{e^2}{M\omega_2^2 x_0^2} \tag{4.12}$$

From (4, 7) it is evident that

$$\partial U/\partial y \geqslant 0, \qquad \partial U/\partial z \geqslant 0, \qquad U(x, y, \pm\infty) \geqslant 0$$

Hence , for $U_1 < E < U_2$ the equipotential surface breaks up into two surfaces, one of which is closed and contains the origin of the coordinate system, while the other contains the minimum point of $U(x, y, z)$ with the coordinates $x = x_m,\ y = z = 0$,

$$x_m = {}^1/_3 r_0 \,[1 + 2\cos {}^1/_3 (\pi - \theta)] \tag{4.13}$$

and is closed if $U_1 < E < 0$ and open for $z = \pm\infty$ if $U_2 > E > 0$, which is possible provided that

$$\frac{2}{27}\,\frac{M\omega_2^2 x_0^2}{e^2} > 1$$

which follows from (4. 10), (4. 12). The case where $U_2 > 0$ means that three-dimensional finite motion about the origin is possible, although the total energy is larger than the minimum of $V(x, y, z)$ for $z = \pm\infty$. For $U_1 < E < 0$, in addition to the ordinary (Keplerian) finite motion about the origin it is also possible to have finite motion about the point $x = x_m$ (4. 13), $y = z = 0$. In considering trajectories it should be borne in mind that inclusion of the term

$$m\omega_3\,(xy^{\cdot} - yx^{\cdot})$$

in Lagrangian (4. 6) naturally imposes an additional limitation on transverse motion across the magnetic field.

Turning back to Equation (4. 5), we can readily show that it determines the coordinates of the maximum and minimum of $U(x, 0, 0)$ in accordance with (4. 8). It has two positive roots if condition (4. 9), where

$$x_0 = -\sqrt{2MP_\tau}\,/\,M\omega_2$$

is fulfilled.

Here x_m (4. 13) is a large root associated with a stable trivial solution. Expressions (4. 9) and (4. 13) likewise imply that

$$x_m > (2\,Mc^2/B^2)^{1/3} \tag{4.14}$$

It should be noted that the uniform rotation of the particles which corresponded to the time-independent solution of system (4. 1) to (4, 4) degenerates with $\omega_1 = 0$ into straight-line motion with constant velocity. This can be illustrated as follows. Let particles with charges of opposite sign $e_1 = -e_2 = -e$ move across a homogeneous magnetic field at the same velocity V in the direction perpendicular to the radius vector connecting these particles and lying entirely in the xy plane.

For any given field and velocity there always exists a radius vector of a length such that the Lorentz force for both particles is balanced by the Coulomb force

$$eBV/c = e^2/\rho^2 \tag{4.15}$$

where ρ is given by Equation (4. 5).

Making use of (4. 14) and (4. 15), we can write the stability condition in the form

$$V < e\left(\frac{B}{4M^2c}\right)^{1/3}$$

From (4. 14) and (4. 16) it follows that as B tends to zero, the distance x_m between a stable pair of particles tends to infinity and their velocity to zero,. Only in sufficiently strong (10^4 - 10^5 G) magnetic fields can the velocity of a stable pair (a positive and negative ion, an electron and an ion or an electron and a hole) become comparable with the mean thermal velocity and the pair size with the free path.

As we see from (4. 6), for $P_{0x} = P_{0y} = 0$ the problem reduces to a plane one by virtue of cylindrical symmetry. For P_{0x}, $P_{0y} \neq 0$ the problem can be reduced to quadratures in the particular plane case $\omega_1 = \omega_3 = 0$, $z^{\cdot} = z = 0$; this solution is obviously stable in z. What we have in this case is a Liouville system [3], and in elliptical coordinates with the origin at the point ($x = x_0$, $y = 0$) the equation of the trajectory is of the form

$$\int_{\zeta_0}^{\zeta} [f(\cosh\zeta)]^{-1/2}\, d\zeta = \int_{\eta_1}^{\eta} [-f(\cos\eta)]^{-1/2}\, d\eta \qquad (4.17)$$

$$f(\cosh\zeta) = -\frac{M\omega_2^2 x_0^2}{2}\cosh^4\zeta + \left(\frac{M\omega_2^2 x_0^2}{2} + \gamma\right)\cosh^2\zeta + \frac{e^2}{x_0}\cosh\zeta + \delta$$

Here γ δ are integration constants. In fact, it is possible here that the motion is along ellipses with foci at the origin and at the point $x = 2x_0$, $y = 0$. This follows from the theorem of Bonnet [3]. It can also be shown that in the three-dimensional case for $E > U_2$ the equipotential surfaces form open traps near the origin. It is interesting in this case to estimate the duration of capture in such a trap of a particle approaching from infinity with $E > 0$. In terms of the two-body problem, capture constitutes the formation of a bound pair of particles lying at infinity $z = \pm\infty$ at $t = \pm\infty$. This problem requires special study and will not be considered here.

We have thus derived integrals of motion and used them to carry out canonical transformations whereby the classical nonrelativistic problem of motion of two interacting charged particles in a homogeneous stationary magnetic field is reduced to a one-particle relative motion problem for arbitrary charges and masses of both particles. We have shown that with the exception of the case where the specific charges are equal, the motion of the center of mass affects the relative motion. In the particular case where the particles have charges of opposite sign but the same absolute value there arise stable equilibrium states not of the ordinary Keplerian type.

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